

Correctness of Multiplicative (and Exponential) Proof Structures is *NL*-Complete

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Abstract. We provide a new correctness criterion for unit-free MLL proof structures and MELL proof structures with units. We prove that deciding the correctness of a MLL and of a MELL proof structure is *NL*-complete. We also prove that deciding the correctness of an intuitionistic multiplicative essential net is *NL*-complete.

Introduction

The *proof nets* [Gir87,DR89] of Linear logic (LL) are a parallel syntax for logical proofs without all the bureaucracy of sequent calculus. They are a non-sequential graph-theoretic representation of proofs, where the order in which some rules are used in a sequent calculus derivation, when irrelevant, is neglected. The unit-free multiplicative proof nets are inductively defined from sequent calculus rules of unit-free Multiplicative Linear Logic (MLL). A *proof structure* is freely built on the same syntax as proof nets, without any reference to a sequent calculus derivation.

In LL we are mainly interested in the following decision problems: Deciding the provability of a given formula, which gives the expressiveness of the logic; deciding if two given proofs reduce to the same normal form, i.e. the cut-elimination problem which corresponds to program equivalence using the Curry-Howard isomorphism; and deciding the correctness of a given proof structure, i.e. whether it comes from a sequent calculus derivation. For this last decision problem, one uses a *correctness criterion* to distinguish proof nets among proof structures. We recall the following main results [Kan92,Mai] and we complete (in bold) the correctness cases:

fragment	decision problem			
	units	provability	cut-elimination	correctness
MLL	no	<i>NP</i> -complete	<i>P</i> -complete	NL-complete
MELL	yes	open	(at most non-elementary)	

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Correctness is equivalent to provability for unit only MLL because proof nets are formulae syntactic trees. However it is not so obvious for the propositional case as one can observe following the long story of correctness criteria:

- Long-trip [Gir87] is based on travels and was the first one.
- Acyclic-Connected [DR89] is a condition is based on switchings i.e. the choice of one premise for each \wp connective. The condition is that all the associated graphs are trees. A naive implementation of Acyclic-Connected uses exponential time.
- Contractibility [Dan90] is done in quadratic time by repeating two graph rewriting rules until one obtains a simple node.
- Graph Parsing [Laf95] is a strategy for Contractibility which is implemented in linear time as a sort of unification [Gue99].
- Dominator Tree [MO00,MO06] is a linear time correctness criterion for essential nets, to which proof structures correctness reduces in linear time.
- Ribbon [Mel04] is a topological condition requiring homeomorphism to the disk.

For other fragments of Linear Logic, some of these criteria apply or are extended as for MELL¹ [Dan90,GM01].

A feature of these criteria is that they successively lower the complexity of sequential, deterministic algorithms deciding correctness for MLL. Switching from proof structures to paired graphs, that is undirected graphs with a distinguished set of edges, we give a new correctness criterion for MLL and more generally for MELL. This new correctness criterion gives us a lower bound for the correctness decision problem for both MLL and MELL. This lower bound yields an exact characterization of the complexity of this problem, and induces naturally efficient parallel and randomized algorithms for it.

The paper is organized as follows: we recall preliminary definitions and results in linear logic and complexity theory in Section 1. Section 2 is devoted to the exposition of a new correctness criterion for unit-free MLL and MELL with units. We prove its *NL*-completeness in Section 3, and the *NL*-completeness of the criterion for IMLL in Section 4.

1 Background

1.1 MLL and Proof Structures

Roman capitals A, B stand for MLL formulae, which are given by the following grammar, where \otimes and \wp are duals for the negation $^\perp$, accordingly to De Morgan laws:

$$F ::= A \mid A^\perp \mid F \otimes F \mid F \wp F$$

¹ As usual M, A and E denote respectively for Multiplicative, Additive and Exponential fragments of LL

Greek capitals Γ, Δ stand for sequents, which are multiset of formulae, so that exchange is implicit. The MLL sequent calculus is given by the following rules:

$$\frac{}{\vdash A, A^\perp} (ax) \quad \frac{\vdash \Gamma, C \quad \vdash \Delta, C^\perp}{\vdash \Gamma, \Delta} (cut) \quad \frac{\vdash \Gamma, A \quad \vdash \Delta, B}{\vdash \Gamma, \Delta, A \otimes B} \otimes \quad \frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \wp B} \wp$$

Definition 1. A MLL proof structure is a finite directed acyclic graph (DAG)² whose nodes, called links, are defined together with an arity and a coarity, i.e. a given number of incident edges called the premises of the node and a given number of emergent edges called the conclusions of the node. Moreover the proof structure edges are labelled by formulae and every edge is conclusion of exactly one link and premise of at most one link. The links of are the following:

nodes		ax		cut		\otimes		\wp	
arity	edge labels	0	\emptyset	2	A, A^\perp	2	A, B	2	A, B
coarity	edge labels	2	A, A^\perp	0	\emptyset	1	$A \otimes B$	1	$A \wp B$

We allow edges with a source but no target (i.e pending or dandling edges), they are called the conclusions of the proof-structure.

A MLL proof net is a MLL proof structure inductively defined as follows:

- an ax-link is a proof net with conclusions A, A^\perp ,
- if P is a proof net with conclusions Γ, A, B then P extended with a \wp -link of premises A and B is a proof net with conclusions $\Gamma, A \wp B$.
- if P_1 and P_2 are disjoint proof nets with respective conclusions Γ, A and Δ, B then $P_1 \cup P_2$ extended with a \otimes -link of premises A and B is a proof net with conclusions $\Gamma, A \otimes B, \Delta$.

It follows from the definition that MLL proof structures and proof nets have a non-empty set of conclusions, which corresponds to a MLL sequent. The inductive definition of MLL proof nets corresponds to a graph theoretic abstraction of the derivation rules of MLL; any proof net is sequentializable, i.e. corresponds to a MLL derivation: given a proof net P of conclusion Γ , there exists a sequent calculus proof of $\vdash \Gamma$ which infers P .

Definition 2. A paired graph is an undirected graph $G = (V, E)$ with a set of pairs $C(G) \subseteq E \times E$ which are pairwise disjoint couples of edges with the same target, called a pair-node, and two (possibly distinct) sources called the premise-nodes.

A switching S of G is the choice of an edge for every pair of $C(G)$. With each switching S is associated a subgraph $S(G)$ of G : for every pair of $C(G)$, erase the edges which are not selected by S . When S selects the (abusively speaking) left edge of each pair, $S(G)$ is denoted as $G[\forall \mapsto \cdot]$. Also, $G[\forall \mapsto \cdot]$ stands for $G \setminus \{e, e' \mid (e, e') \in C(G)\}$.

² For convenience the edges are oriented up-down, so we do not mention the orientation

Let $R = (V, E)$ be a MLL proof structure. To R , we naturally associate the paired graph $G_R = (V, E')$ where E' is the set of non-pending edges of E and $C(G_R)$ contains the premises of each \wp -link of R (Figure 1). For a pair of edges $(v, x), (w, x)$, we adopt the representation of Figure 1, where the two edges of the pair are joined by an arc.

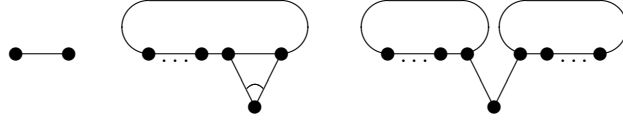


Fig. 1. Paired graph constructors associated to MLL proof nets: ax -link, \wp -link and \otimes -link.

We define the following graph rewriting rules \rightarrow_c of Figure 2 on paired graphs where all the nodes are distinct and rule \rightarrow_{R_2} applies only for a non-pair edge:

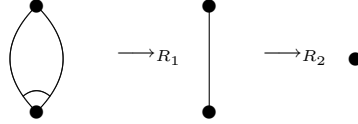


Fig. 2. Contraction rules \rightarrow_c

We denote by $G \rightarrow_c \bullet$ the fact that G contracts to a single vertex with no edge.

Definition 3. A MLL proof structure R is DR-correct if for all switching S of G_R , the graph $S(G_R)$ is acyclic and connected.

A MLL proof structure R is contractile if $G_R \rightarrow_c^* \bullet$.

Theorem 1. [DR89,Dan90] A MLL proof structure R is a MLL proof net iff R is DR-correct iff R is contractile³.

We define the following decision problem MLL-CORR:

GIVEN: A MLL proof structure R

PROBLEM: Is R a MLL proof net?

1.2 MELL and Proof Structures

The definition of MELL formulae follows that of MLL formulae in Section 1.1, with $!$ and $?$ duals for the negation \perp , as well as the neutral elements 1 and \perp :

$$\text{MELL:} \quad F ::= A \mid A^\perp \mid F \otimes F \mid F \wp F \mid !F \mid ?F \mid 1 \mid \perp$$

The MELL sequent calculus contains the rules of the MLL sequent calculus, as well as the following rules:

$$\frac{\vdash \Gamma}{\vdash \Gamma, \perp} \perp \quad \frac{}{\vdash 1} 1 \quad \frac{\vdash \Gamma}{\vdash \Gamma, ?A} ?W \quad \frac{\vdash \Gamma, ?A, ?A}{\vdash \Gamma, ?A} ?C \quad \frac{\vdash \Gamma, A}{\vdash \Gamma, ?A} ?D \quad \frac{\vdash ?\Gamma, A}{\vdash ?\Gamma, !A} !P$$

³ The criteria in [DR89,Dan90] are expressed for switchings and contraction rules for proof structures only. The equivalence with Definition 3 is left to the reader.

Definition 4. MELL structures are defined similarly to MLL proof structures (Definition 1), with the additional links, where the $?W$ -link subsumes both $?W$ and \perp rules:

nodes		1		$?W$	$?C$	$?D$	$!P$
arity	edge labels	0	\emptyset	\emptyset	2 $?A, ?A$	1 A	1 A
coarity	edge labels	1	1	$1 \perp$ or $?A$	1 $?A$	1 $?A$	1 $!A$

Definition 5. An exponential box is a MELL structure whose conclusions are all $?$ -formulae but one, its principal door, which is conclusion of a $!P$ -link. Similarly, a weakening box is a MELL structure with a distinguished conclusion, its principal door, which is conclusion of a $?W$ -link. A box is either an exponential or a weakening box.

Definition 6. A MELL proof structure (R, B) is given by a MELL structure R and a box mapping B , which associates to any link l of R a box b_l or R . Moreover, boxes may nest but may not partially overlap, and a unique exponential (respectively weakening) box is associated to each $!P$ -(resp. $?W$)-link. By convention, when a link belongs to several boxes, the mapping returns the innermost box to which it belongs, otherwise it returns R .

It follows from the definition that, for any $!P$ (respectively $?W$)-link, the box mapping associates the exponential (resp. weakening) box to which it naturally corresponds. The whole proof structure R is treated as a particular box, and is associated to all links that do not belong to any exponential or weakening box. Let (R, B) be a MELL proof structure, with boxes b_1, \dots, b_n . Let $b_0 = R$. We define as follows the family $\mathcal{G}_{(R,B)} = \{G_{(R,B)}^i\}_{i=0\dots n}$ of paired-graphs:

- $G_{(R,B)}^i$ contains a node l for every link l of $R \setminus \{?W\text{-links}\}$ with $B(l) = b_i$, and an edge (l, l') for all links l, l' of $R \setminus \{?W\text{-links}\}$ with $B(l) = B(l') = b_i$. $C(G_{(R,B)}^i)$ contains the premises of each \wp -link and $?C$ -link l of R with $B(l) = b_i$.
- Assume b_j is an outermost box included in b_i . A node $\overline{b_j} \in G_{(R,B)}^i$ is associated to b_j , and an edge $(\overline{b_j}, l) \in G_{(R,B)}^i$ for all link l conclusion of a link in b_j and such that $B(l) = b_i$.

Essentially, $G_{(R,B)}^i$ is the paired graph corresponding to the box b_i , where all inner boxes are considered contracted to a single node. Moreover, for the sake of connectivity, the $?W$ -link (if there is any) corresponding to b_i is removed.

Definition 7. A MELL proof structure (R, B) with boxes b_1, \dots, b_n is contractile if $\forall i \in \{0, \dots, n\}$, $G_{(R,B)}^i \rightarrow_c^* \bullet$.

As for MLL, one may inductively define particular MELL proof structures, called MELL proof nets, which exactly correspond to derivations in MELL sequent calculus. Theorem 2 allows to distinguish MELL proof nets among proof structures:

Theorem 2. [GM01] A MELL proof structure (R, B) is a MELL proof net iff (R, B) is contractile⁴.

⁴ The criterion in [GM01] uses contraction rules for MELL proof structures only. As for Theorem 1, the proof of the equivalence with Definition 7 is left to the reader.

We define the following decision problem MELL-CORR:

GIVEN: A MELL proof structure (R, B)

PROBLEM: Is (R, B) a MELL proof net?

1.3 Intuitionistic Multiplicative Linear Logic and Essential Nets

The intuitionistic fragment of MLL (IMLL) is the (\otimes, \multimap) -fragment of Linear logic, where linear implication is no more defined by $A \multimap B = A^\perp \wp B$ but is a connective. The sequent calculus corresponds to the MLL sequent calculus but with two-sided sequents (using linear negation) and at most one (distinguished) formula on the right.

For this sub-logic, Lamarche has proposed a version of proof structures called *essential nets*. They are built on the links given in Figure 3 where, to each \wp^+ -labelled node, one associates a negatively-labelled node (left premise) called the sink of p . They also have a distinguished link called the *root*.

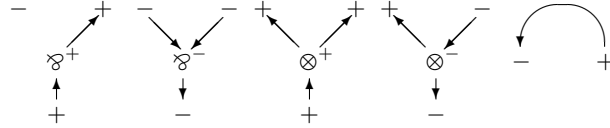


Fig. 3. Essential net links: \wp^+ , \wp^- , \otimes^+ , \otimes^- and Axiom

Definition 8. An essential net of a linearly balanced⁵ IMLL sequent is *L-correct* if it is acyclic, every node is reachable from the root, and every \wp^+ -node p satisfies the *L-condition*: every path from the root that reaches the sink of p passes through p .

Lamarche has shown that the essential net of an IMLL sequent denotes a IMLL sequent derivation if and only if it is *L-correct*. For additional information on essential net correction, including translation to proof structures, we refer the reader to [MO06].

We define the following decision problem IMLL-CORR:

GIVEN: A multiplicative essential net N of a linearly balanced IMLL sequent

PROBLEM: Is N correct?

1.4 Complexity Classes and Related Problems

We will mention several major complexity classes below P , some of which having natural complete problems that we will use in this paper. Let us briefly recall some basic definitions and results.

Definition 9. *Complexity classes:*

⁵ I.e. every atom that occurs in the sequent does so exactly twice, once positively and once negatively

- AC^0 is the class of problems solvable by a uniform family of circuits of constant depth and polynomial size, with NOT gates and AND, OR gates of unbounded fan-in.
- NC^1 is the class of problems solvable by a uniform family of circuits of logarithmic depth and polynomial size, with NOT, AND and OR gates of bounded fan-in.
- L is the class of problems solvable by a deterministic Turing machine which only uses a logarithmic working space.
- NL is the class of problems solvable by a non-deterministic Turing machine which only uses a logarithmic working space, such that:
 1. If the answer is "yes," at least one computation path accepts.
 2. If the answer is "no," all computation paths reject.
- $coNL$ is the class of problems solvable by a non-deterministic Turing machine which only uses a logarithmic working space, such that:
 1. If the answer is "yes," all computation paths accept.
 2. If the answer is "no," at least one computation path rejects.
- AC^1 is the class of problems solvable by a uniform family of circuits of logarithmic depth and polynomial size, with NOT gates and AND, OR gates of unbounded fan-in.

Theorem 3. [Imm88,Sze87] $NL = coNL$.

The following inclusion results are also well known:

$$AC^0 \subset NC^1 \subset L \subset NL \subset AC^1 \subset P, \quad (1)$$

where it remains unknown whether any of these inclusions is strict.

It is important to note that Theorems 6 and 7 give NL -completeness results under constant-depth (actually AC^0) reductions. From (1) above, it should be clear to the reader that the reductions lie indeed in a class small enough for being relevant. For a good exposition of constant-depth reducibility, we refer the reader to [CSV84].

In the sequel, we will often use the notion of a *path* in a directed -or undirected- graph. A path is a sequence of vertices such that there is an edge between any two consecutive vertices in the path. A path will be called *elementary* when any node occurs at most once in the path.

Let us now list some graph-theoretic problems that will be used in this paper.

IS TREE (IT)

GIVEN: An undirected graph $G = (V, E)$

PROBLEM: Determine whether G is a tree

IT is L -complete under constant-depth reductions [JLM97].

SOURCE-TARGET CONNECTIVITY (STCONN)

GIVEN: A directed graph $G = (V, E)$ and two designated vertices s and t

PROBLEM: Is there a path from s to t in G ?

STCONN is NL -complete under constant-depth reductions [JLL76].

UNDIRECTED SOURCE-TARGET CONNECTIVITY (USTCONN)

GIVEN: An undirected graph $G = (V, E)$ and two designated vertices s and t

PROBLEM: Do s and t belong to the same connected component of G ?

USTCONN is L -complete under constant-depth reductions [Rei05].

UNIVERSAL SOURCE DAG (SDAG)

GIVEN: A directed graph $G = (V, E)$ and a designated (source) vertex s

PROBLEM: Is G acyclic and is there a path from the source s to each vertex ?

Proposition 1. $\text{SDAG} \in NL$.

Proof. Given $G = (V, E)$ a directed graph, its acyclicity can be expressed as follows:

$$\forall (x, y) \in V^2 : \neg \text{STCONN}(G, x, y) \vee \neg \text{STCONN}(G, y, x).$$

Since $NL = coNL$ (Theorem 3) and $\text{STCONN} \in NL$, acyclicity is clearly in NL . Checking whether each vertex can be reached from the source vertex s can also be done with STCONN subroutines, therefore SDAG is in NL . \square

Proposition 2. SDAG is $coNL$ -hard under constant-depth reductions.

Proof. Let \mathcal{L} be any language in $coNL$. \mathcal{L} is then decided by a non-deterministic Turing machine M in space less than $k \log(n)$ on inputs of size n , for some $k \geq 0$. Let \mathcal{C}_n be the set of configurations of M of size less or equal to $k \log(n)$, and define $T = |\mathcal{C}_n|$. Clearly, $T \leq n^k$ is an upper bound for the computation time of M on inputs of size n . Without loss of generality, we assume that every configuration of M has at least one outgoing transition, possibly towards itself, and that the result of the computation is given by the state reached by M after exactly T computation steps. A configuration is thus either accepting or rejecting.

Let us consider the following directed graph $G_n = (V_n, E_n)$, where:

$$V_n = \bigcup_{c \in \mathcal{C}_n, t \in [0, T]} \{(c, t)\} \cup \{c_A\} \cup \{c_R\} \cup \{s\}.$$

For $(c, t), (c', t+1) \in V_n$, $((c', t+1) \rightarrow (c, t)) \in E_n$ if and only if $c \rightarrow c'$ is a transition of M .

For $c \in \mathcal{C}_n$, $(c_A \rightarrow (c, T)) \in E_n$ iff c is an accepting configuration of M .

For $c \in \mathcal{C}_n$, $(c_R \rightarrow (c, T)) \in E_n$ iff c is a rejecting configuration of M .

$(s \rightarrow c_A) \in E_n, (s \rightarrow c_R) \in E_n$.

A path $(c_1, t_1) \rightarrow \dots \rightarrow (c_k, t_k)$ in G_n follows by construction a sequence t_1, \dots, t_k that is strictly decreasing. Since there is no edge $(c, t) \rightarrow c_A$, $(c, t) \rightarrow c_R$ nor $(c, t) \rightarrow s$, it is then clear that G_n is acyclic.

Moreover, since every configuration of M has at least one outgoing transition, every vertex $(c, t), t < T$ in G_n has at least one parent node $(c', t+1)$. By induction on t , it follows that every vertex in G_n is reachable from s . Therefore G_n satisfies SDAG .

Let x be an input of size n to M . An initial configuration $c_x \in \mathcal{C}_n$ of M is naturally associated to this input x . Consider now the directed graph $H_n^x = G_n \cup \{(c_x, 0) \rightarrow c_R\}$.

Then, H_n^x satisfies SDAG if and only if $x \in \mathcal{L}$. Indeed, by Definition 9, $x \in \mathcal{L}$ if and only if there exists no computation path $c_x \rightarrow \dots \rightarrow c_r$ of length T in M , where c_r is a rejecting configuration. By construction of G_n , such a path corresponds to a path $(c_r, T) \rightarrow \dots \rightarrow (c_x, 0)$ in G_n . Then $x \in \mathcal{L}$ if and only if there exists no path $c_R \rightarrow \dots \rightarrow (c_x, 0)$ in G_n , if and only if H_n^x is acyclic. Since G_n satisfies SDAG, it follows that H_n^x satisfies SDAG if and only if $x \in \mathcal{L}$. Moreover, it is well known that the configuration graph of a Turing machine can be computed with constant-depth circuits. Computing H_n^x from the configuration graph of M requires only purely local rewriting rules, that can all be performed in parallel. Therefore, H_n^x can also be computed with constant-depth circuits. \square

Propositions 1 and 2, and Theorem 3 yield the following result:

Theorem 4. *SDAG is NL-complete under constant-depth reductions.*

2 New Correctness Criteria for MLL and MELL

Definition 10. *Let G be a paired graph. The dependency graph $D(G)$ of G is the directed graph (V_G, E_G) defined as follows:*

- $V_G = \{v \mid v \text{ is a pair-node in } G\} \cup \{s\}$.
- Let x be a pair-node in G , with premise-nodes x_l and x_r . The edge $(s \rightarrow x)$ is in E_G if and only if:
 1. There exists an elementary path $p_x = x_l, \dots, x_r$ in $G[\forall \mapsto \lambda \cdot]$,
 2. $x \notin p_x$, and
 3. for all pair-node y in G , $y \notin p_x$.
- Let x be a pair-node in G , with premise-nodes x_l and x_r , and let $y \neq x$ be another pair-node in G . The edge $(y \rightarrow x)$ is in E_G if and only if:
 1. There exists an elementary path $p_x = x_l, \dots, x_r$ in $G[\forall \mapsto \lambda \cdot]$,
 2. $x \notin p_x$, and
 3. for every elementary path $p_x = x_l, \dots, x_r$ in $G[\forall \mapsto \lambda \cdot]$ with $x \notin p_x$, $y \in p_x$.

Lemma 1. *Let G and H be paired graphs, with $G \rightarrow_c H$. Then, $G[\forall \mapsto \lambda \cdot] \rightarrow_c^* H[\forall \mapsto \lambda \cdot]$, and $G[\forall \mapsto \lambda \cdot]$ is a tree if and only if $H[\forall \mapsto \lambda \cdot]$ is a tree.*

Proof. If $G \rightarrow_{R_1} H$ denote by v the redex pair-node in G , with premise w . The reduced pattern in H is the non-pair edge (v, w) , therefore $G[\forall \mapsto \lambda \cdot] = H[\forall \mapsto \lambda \cdot]$. If $G \rightarrow_{R_2} H$, it is clear that $G[\forall \mapsto \lambda \cdot] \rightarrow_{R_2} H[\forall \mapsto \lambda \cdot]$ with the same redex. It is also clear that rule \rightarrow_{R_2} preserves connectivity and acyclicity. \square

Lemma 2. *If $G \rightarrow_c^* \bullet$ then $D(G)$ satisfies SDAG.*

Proof. Since $\bullet[\forall \mapsto \lambda \cdot]$ is a tree, by Lemma 1 so is $G[\forall \mapsto \lambda \cdot]$. Therefore, for any pair-node x with premise-nodes x_l and x_r in G , there exists a unique elementary path $p_x = x_l \dots x_r$ in $G[\forall \mapsto \lambda \cdot]$. It follows by construction of $D(G)$ that x has at least one parent node in $D(G)$. Moreover, a path $x \rightarrow \dots \rightarrow y$ in $D(G)$ induces by construction an elementary path $x_l \dots y$ in $G[\forall \mapsto \lambda \cdot]$. Therefore a cycle $x \rightarrow \dots \rightarrow y, y \rightarrow \dots \rightarrow x$ in $D(G)$ induces a cycle $x_l \dots y, y_l \dots x$ in $G[\forall \mapsto \lambda \cdot]$. Since $G[\forall \mapsto \lambda \cdot]$ is a tree, $D(G)$ is acyclic. Since every vertex of $D(G)$ but s has at least one parent node and $D(G)$ is acyclic, $D(G)$ satisfies SDAG. \square

Lemma 3. *Let G be a paired graph such that $G[\forall \mapsto \lambda \cdot]$ is a tree. If the dependency graph $D(G)$ of G satisfies SDAG then $G \rightarrow_c^* \bullet$.*

Proof. let $d(v)$, the depth of a pair-node $v \in G$, be the length of the longest path from the source s of $D(G)$ to the vertex $v \in D(G)$. Assume that $D(G)$ satisfies SDAG, and let $X^d = \{x \text{ pair-node in } G \mid d(x) = d\}$ and $Y^d = \cup_{d' \leq d} X^{d'}$. By induction on the depth we prove that there exists a sequence of contractions \mathcal{C}_d such that $G \rightarrow_{\mathcal{C}_d} G^d$ satisfies:

$$\text{Each pair-node } y \in G \text{ s.t. } d(y) \leq d \text{ is contracted in } G^d. \quad (2)$$

The proof by induction is as follows:

- For $d = 1$, let $x \in X^1$, with premise-nodes x_l and x_r . By definition of X^1 , there exists an elementary path $p_x = x_l - \dots - x_r$ in $G[\forall \mapsto \lambda \cdot]$ such that $x \notin p_x$ and for any pair-node y in $G[\forall \mapsto \lambda \cdot]$, $y \notin p_x$. The same holds for the path $p_x = x_l - \dots - x_r$ in G , with respect to any pair-node $y \in G$.
Let $\mathcal{E}_x^1 = \{e \text{ edge of } p_x \mid x \in X^1\}$. The set of contractions $\mathcal{R}_x^1 = \{e \rightarrow_c \bullet \mid e \in \mathcal{E}_x^1\}$ contracts the edges of p_x , and let $\mathcal{R}^1 = \cup_{x \in X^1} \mathcal{R}_x^1$. Clearly, $x_l = x_r \neq x$ in the contracted paired graph obtained from G by \mathcal{R}_x^1 . Since $x \notin p_y$ for any $y \in X^1$, the same holds for the contracted paired graph obtained from G by \mathcal{R}^1 .
Let \mathcal{C}_1 be the sequence \mathcal{R}^1 , followed by the set of contraction rules of the pair-nodes $x \in X^1$. Define G^1 such that $G \rightarrow_{\mathcal{C}_1} G^1$. It is clear that G^1 satisfies (2).
- Assume by induction that there exists a sequence of contractions \mathcal{C}_d such that $G \rightarrow_{\mathcal{C}_d} G^d$ satisfies (2).
Let $x \in X^{d+1}$, with premise-nodes x_l and x_r .
Since $G \rightarrow_{\mathcal{C}_d} G^d$ and $G[\forall \mapsto \lambda \cdot]$ is a tree, Lemma 1 applies:

$$G[\forall \mapsto \lambda \cdot] \rightarrow_{\mathcal{C}_d} G^d[\forall \mapsto \lambda \cdot], \text{ and } G^d[\forall \mapsto \lambda \cdot] \text{ is a tree.} \quad (3)$$

By definition of X^{d+1} , there exists an elementary path $p_x = x_l - \dots - x_r$ in $G[\forall \mapsto \lambda \cdot]$ such that $x \notin p_x$ and, for every pair-node $y \in G$ of depth $d(y) > d$, $y \notin p_x$.

Define p_x^d such that $p_x \rightarrow_{\mathcal{C}_d} p_x^d$. By (3), p_x^d is an elementary path in $G^d[\forall \mapsto \lambda \cdot]$ such that $x \notin p_x^d$ and, for every pair-node $y \in G^d[\forall \mapsto \lambda \cdot]$ of depth $d(y) > d$, $y \notin p_x^d$. The same holds for p_x^d in G^d , with respect to any pair-node $y \in G^d$, since, by induction, for any pair-node $y \in G^d$, $d(y) > d$.

Let $\mathcal{E}_x^{d+1} = \{e \text{ edge of } p_x \mid x \in X^{d+1}\}$. The set of contractions $\mathcal{R}_x^{d+1} = \{e \rightarrow_c \bullet \mid e \in \mathcal{E}_x^{d+1}\}$ contracts the edges of p_x^d , and let $\mathcal{R}^{d+1} = \cup_{x \in X^{d+1}} \mathcal{R}_x^{d+1}$. Clearly, $x_l = x_r \neq x$ in the contracted paired graph obtained from G by \mathcal{R}_x^{d+1} . Since $x \notin p_y$ for any $y \in X^{d+1}$, the same holds for the contracted paired graph obtained from G by \mathcal{R}^{d+1} .

Let \mathcal{C}_{d+1} be the sequence \mathcal{C}_d , followed by \mathcal{R}_{d+1} , and followed by the set of contraction rules of the pair-nodes $x \in X^{d+1}$. Define G^{d+1} such that $G \rightarrow_{\mathcal{C}_{d+1}} G^{d+1}$. G^{d+1} satisfies (2).

Since $D(G)$ satisfies SDAG, the maximal depth $m = \max\{d(x) | x \in D(G)\}$ is well-defined and every pair-node x of G belongs to X^m . Therefore, $G \rightarrow_c^m G^m$ and G^m satisfies (2). Since $G[\forall \mapsto \lambda \cdot]$ is a tree, by Lemma 1 so is $G^m[\forall \mapsto \lambda \cdot] = G^m$. It follows that $G \rightarrow_c^* \bullet$. \square

Lemmas 2 and 3 and Theorems 1 and 2 imply the Theorem:

Theorem 5 (Correctness Criteria).

A MLL proof structure R is a MLL proof net if and only if:

1. $D(G_R)$ satisfies SDAG, and
2. $G_R[\forall \mapsto \lambda \cdot]$ is a tree.

A MELL proof structure (R, B) with boxes b_1, \dots, b_n is a MELL proof net if and only if:

1. $\forall i \in \{0, \dots, n\}$, $D(G_{(R,B)}^i)$ satisfies SDAG, and
2. $\forall i \in \{0, \dots, n\}$, $G_{(R,B)}^i[\forall \mapsto \lambda \cdot]$ is a tree.

3 NL-Completeness of the Criteria for MLL and MELL

Let DEPGRAPH be the function: $G \mapsto D(G)$, which associates its dependency graph to a paired graph G .

Lemma 4. DEPGRAPH $\in FL$.

Proof. The following functions can easily be computed in FL:

- $G, x \in G \mapsto (G[\forall \mapsto \lambda \cdot]) \setminus \{x\}$
- $G, x \in G \mapsto (G[\forall \mapsto \cdot]) \setminus \{x\}$
- $G, x \in G, y \in G \mapsto (G[\forall \mapsto \lambda \cdot]) \setminus \{x, y\}$

Consider now the following algorithm for DEPGRAPH:

```

INPUT ( $G$ )
FOR ALL  $x$  pair-node in  $G$ , with premise-nodes  $x_l$  and  $x_r$  DO
  IF USTCONN( $(G[\forall \mapsto \cdot]) \setminus \{x\}, x_l, x_r$ ) THEN OUTPUT  $(s \rightarrow x) \in D(G)$ 
FOR ALL ( $x$  pair-node in  $G$ , with premise-nodes  $x_l$  and  $x_r$ ,  $y$  pair-node in  $G$ ) DO
  IF  $\neg$ USTCONN( $(G[\forall \mapsto \lambda \cdot]) \setminus \{x, y\}, x_l, x_r$ )
  AND USTCONN( $(G[\forall \mapsto \lambda \cdot]) \setminus \{x\}, x_l, x_r$ ) THEN
    OUTPUT  $(y \rightarrow \{x\}) \in D(G)$ .

– USTCONN( $(G[\forall \mapsto \cdot]) \setminus \{x\}, x_l, x_r$ ) tests whether there exists an elementary path  $p_x = x_l - \dots - x_r$  such that  $x \notin p_x$  and, for all pair-node  $y$  in  $G$ ,  $y \notin p_x$ .
–  $\neg$  USTCONN( $(G[\forall \mapsto \lambda \cdot]) \setminus \{x, y\}, x_l, x_r$ ) tests whether any elementary path  $p_x = x_l - \dots - x_r$  such that  $x \notin p_x$  contains  $y$ .
– USTCONN( $(G[\forall \mapsto \lambda \cdot]) \setminus \{x\}, x_l, x_r$ ) tests whether there exists a path  $p_x = x_l - \dots - x_r$  in  $G'$  such that  $x \notin p_x$ . From the previous point, if such a path  $p_x$  exists,  $y \in p_x$ .

```

It follows that this algorithm computes DEPGRAPH. Since USTCONN $\in L$, this algorithm belongs to FL^L (the class of functions computable in logarithmic space, with oracles in L). Since $FL^L = FL$, DEPGRAPH $\in FL$. \square

Proposition 3. $MELL - CORR \in NL$.

Proof. Let (R, B) be a MELL-proof structure with boxes b_1, \dots, b_n . Each function $(R, B), i \in \{0, \dots, n\} \mapsto G_{(R, B)}^i$ can be easily be computed in FL . Checking that $G_{(R, B)}^i[\forall \mapsto \cdot]$ is a tree is doable in L since $IT \in L$. Checking that $D(G_{(R, B)}^i)$ satisfies SDAG can be done in NL , by composing the function DEPGRAPH in FL (Lemma 4) with an NL algorithm for SDAG (Theorem 4). Since the number of paired graphs $G_{(R, B)}^i$ is linearly bounded, it suffices to sequentially perform these tasks for $i = 0, \dots, n$, with a counter i of logarithmic size. \square

Proposition 4. $MLL-CORR$ is NL -hard under constant-depth reductions.

Proof. We actually reduce SDAG to MLL-CORR. Let G be a directed graph with a distinguished vertex, and consider the proof structure S_G defined as follows (see Figure 4), and let G_{S_G} be its associated paired graph:

1. To any vertex v of G , we associate a \otimes -link \bar{v} with parent links \bar{v}_{in} and \bar{v}_{out} .
2. If there are $i > 0$ in-going edges to v , \bar{v}_{in} is a \wp -link of arity i , with parent links $\bar{v}_{in}^1, \dots, \bar{v}_{in}^i$. If v has no in-going edge, \bar{v}_{in} is one conclusion of an axiom-link Ax_{in}^v , the other conclusion of Ax_{in}^v being a conclusion of S_n^x .
3. If there are $j > 0$ outgoing edges from v , \bar{v}_{out} is a \otimes -link of arity j , with parent links $\bar{v}_{out}^1, \dots, \bar{v}_{out}^j$. If v has no outgoing edge, \bar{v}_{out} is one conclusion of an axiom-link Ax_{out}^v , the other conclusion of Ax_{out}^v being a conclusion of S_n^x .
4. To an edge $v \rightarrow w$ of G , we associate an axiom-link $Ax^{(v \rightarrow w)}$ with conclusions $Ax_{in}^{(v \rightarrow w)}$ and $Ax_{out}^{(v \rightarrow w)}$. Moreover, if $v \rightarrow w$ is the k^{th} outgoing edge from v , $Ax_{in}^{(v \rightarrow w)}$ is \bar{v}_{out}^k . If $v \rightarrow w$ is the l^{th} in-going edge to w , $Ax_{out}^{(v \rightarrow w)}$ is \bar{w}_{in}^l .

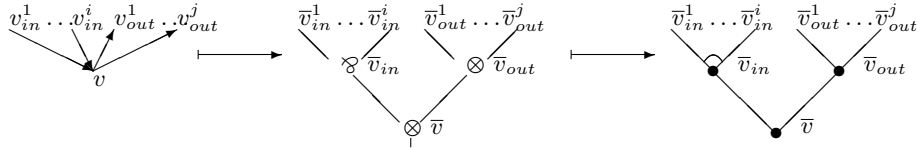


Fig. 4. Construction of S_G and G_{S_G}

It is quite clear that this reduction can be computed by constant-depth circuits. We now claim that S_G is correct if and only if G satisfies SDAG.

Assume G contains a cycle. There exists then an elementary path $p = x_1 \rightarrow \dots \rightarrow x_l$, with $x_l \rightarrow x_1 \in G$. Then, for any edge $x_t \rightarrow x_{t+1} \in p$, there exists a switching of the pair-node \bar{x}_{t+1}_{in} in G_{S_G} , which connects \bar{x}_t and \bar{x}_{t+1} . Similarly for the edge $x_l \rightarrow x_1 \in G$. Since p is elementary, these pair-nodes are all different; therefore there exists cyclic switching of G_{S_G} and S_G is not correct.

Assume that G is acyclic and has no unique source node. Then, there exist at least two nodes, r and s , of arity 0. Let S' be any switching of G_{S_G} , and assume

that there exists an elementary path p from \bar{r} to \bar{s} in S' . Let $p' = \bar{r}, \bar{x}_1, \dots, \bar{x}_k, \bar{s}$ be the sequence of non pair-nodes of p corresponding to vertices of G . p' follows by construction edges of G , accordingly to their orientation or not. Since r and s have arity 0, there exist three nodes $\bar{x}_t, \bar{x}_{t+1}, \bar{x}_{t+2}$ in p' such that $(x_t \rightarrow x_{t+1})$ and $(x_{t+2} \rightarrow x_{t+1})$ are edges of G . By construction of G_{S_G} , \bar{x}_t and \bar{x}_{t+2} are then premise-nodes of the same pair-node \bar{x}_{t+1in} in G_{S_G} , which contradicts that p is a path in S' . Therefore, S' is not connected, and S_G is not correct.

Assume now that G satisfies SDAG and let $d(v)$, the depth of a vertex v of G , be the length of the longest path from the source s of G to v . Denote by G^d the subgraph of G consisting only in the vertices of depth less than d , and by $G_{S_G^d}$ the corresponding paired graph. It is easy to see that the rules of Figure 1 can be turned in a n -ary version, and that $G_{S_G^{d+1}}$ can be obtained from $G_{S_G^d}$ by these n -ary rules. By induction on d , it follows that S_G is correct. \square

Since MLL is a subsystem of MELL, Propositions 3 and 4 immediately yield the following result:

Theorem 6. *MLL-CORR and MELL-CORR are NL-complete under constant-depth reductions.*

Remark 1. In the definition of the dependency graph, one may also choose to consider not only elementary paths, but *any* path. Surprisingly enough, the result holds also for this relaxed version of dependency, even though it is then less clear what dependency means.

4 NL-Completeness of the Criterion for IMLL

Proposition 5. *IMLL – CORR \in NL.*

Proof. For a given essential net N , denote by $r(N)$ its root. For a given \wp^+ -link x in N , denote by $s(x)$ its sink. Consider now the following algorithm for IMLL-CORR:

```

INPUT ( $N$ )
IF  $\neg$ SDAG( $N, r(N)$ ) THEN REJECT
FOR ALL  $x$   $\wp^+$ -link in  $N$  DO
    IF STCONN( $N \setminus \{x\}, r(N), s(x)$ ) THEN REJECT
ELSE ACCEPT.
```

This algorithm checks that N satisfies SDAG, and that the L -condition applies. Since SDAG \in NL and STCONN \in NL, and NL = coNL (Theorem 3), this algorithm belongs clearly to NL. \square

Proposition 6. *IMLL-CORR is NL-hard under constant-depth reductions.*

Proof. We actually reduce SDAG to IMLL-CORR. Let G be a directed graph with a distinguished vertex s , and consider the essential net N_G defined as follows (see Figure 5):

1. To any vertex v of G of arity $i > 0$, we associate a \otimes^- -node \bar{v} , with parent node (right premise) \bar{v}_{in} of polarity $-$ and child node (left premise) \bar{v}_{out} of polarity $+$. To v of arity 0 we associate a node $\bar{v} = \bar{v}_{out}$ of polarity $+$.
2. If there are $i > 0$ in-going edges to v , \bar{v}_{in} is a \wp^- -node of arity i , with parent nodes $\bar{v}_{in}^1, \dots, \bar{v}_{in}^i$ of polarity $-$.
3. If there are $j > 0$ outgoing edges from v , \bar{v}_{out} is a \otimes^+ -node of arity j , with child nodes $\bar{v}_{out}^1, \dots, \bar{v}_{out}^j$ of polarity $+$.
4. If there is no outgoing edge from v , \bar{v}_{out} is a \wp^+ -node with child node (right premise) \bar{v}_{out}^1 of polarity $+$, and sink node (left premise) \bar{v}_{out}^{sink} , of polarity $-$. There is moreover an axiom-edge $\bar{v}_{out}^1 \rightarrow \bar{v}_{out}^{sink}$.
5. Let $v \rightarrow w$ be an edge of G . Assume $v \rightarrow w$ is the k^{th} outgoing edge from v , and the l^{th} in-going edge to w . To $v \rightarrow w$, we associate an axiom-edge from \bar{v}_{out}^k of polarity $+$ to \bar{w}_{in}^l of polarity $-$.

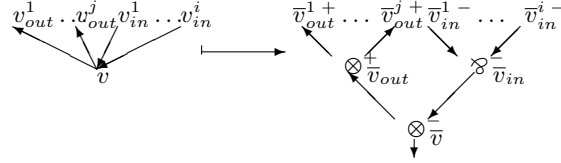


Fig. 5. Construction of N_G

Moreover, let \bar{s} be the root of N_G . It is clear that the reduction is constant-depth. Since the only \wp^+ -links of N_G correspond to leaves of G , N_G satisfies the L -condition by construction. Therefore, it is L -correct if and only if G satisfies SDAG. \square

Propositions 5 and 6 immediately yield the following result:

Theorem 7. *IMLL-CORR is NL-complete under constant-depth reductions.*

Conclusion

Deciding the correctness of unit-free MLL proof structures, MELL proof structures, and unit-free IMLL essential nets were problems known to be decidable in deterministic, sequential linear time. We have shown their NL -completeness, thus establishing that it would be most unlikely to find better sequential deterministic algorithms. As a byproduct, we obtain efficient parallel algorithms for both problems, namely AC^1 algorithms. Moreover, since $NL = RL = ZPL$, we also obtain Monte-Carlo and Las-Vegas logarithmic space random algorithms for both problems. It remains to be checked whether our approach can be extended to MALL.

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